LINEAR ALGEBRA
MID SEMESTRAL EXAM
M MATH - I
2011-2012

Max Score: 100
Time: 3 hours

Answer question 1 and **any five** from the rest.

(1) Justify the following statements.
(i) If A is a 3×3 orthogonal matrix whose determinant is -1, then -1 is an eigenvalue of A.
(ii) Determinant of a hermitian matrix is a real number.

(iii) Let $F = \mathbb{F}_2$, and let $A = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. The bilinear form $X^t A Y$ on $\mathbb{F}_2$ cannot be diagonalized.
(iv) A map $m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a rigid motion fixing origin iff m is left multiplication by an orthogonal matrix.
(v) The only complex matrix which is positive definite, hermitian and unitary is the identity matrix. (5+5+5+5+5)

(2) (a) Let V be a complex vector space of dimension n. Prove that V has dimension $2n$ as a real vector space.
(b) Let W be the space of $n \times n$ real matrices whose trace is zero. Find a subspace W' so that $\mathbb{R}^{n \times n} = W \oplus W'$.
(c) Show that complex $n \times n$ hermitian matrices form a real vector space. Find a basis and determine its dimension. (4+5+6)

(3) (a) Find a basis of the null space of $\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & -6 & 1 & 0 \\ 3 & -5 & 2 & 1 \\ 5 & -4 & 3 & 2 \end{pmatrix}$.
(b) Show that a linear operator T on a finite dimensional vector space V is diagonalizable if and only if the minimal polynomial of T is a product of distinct linear factors. (5+10)

(4) (a) Prove that if the columns of an $n \times n$ matrix A form an orthonormal basis, then the rows of A also do so.
(b) Let V denote the vector space of real $n \times n$ matrices. Prove that $< A, B >= \text{trace}(A^t B)$ is a positive definite bilinear form on V, and find an orthonormal basis for this form. (5+10)

(5) Let $V = \mathbb{R}^{2 \times 2}$ be the space of 2×2 matrices.
(a) Show that the form $< A, B >= \det(A + B) - \det(A) - \det(B)$ is symmetric and bilinear.
(b) Compute the matrix of this form with respect to the standard basis $\{e_{ij}\}$, and determine the signature of the form.
(c) Do the same for the subspace of matrices of trace zero. (4+5+6)

(6) (a) State spectral theorem for a symmetric operator on a real vector space with a positive definite bilinear form.

(b) State the matrix analogue of this theorem.

(c) If $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, find a real orthogonal matrix P so that PAP^t is diagonal. (3+2+10)

(7) (a) State spectral theorem for complex normal matrices.
(b) Let V be a finite dimensional complex vector space with a positive definite hermitian form $<,>$. A linear operator $T : V \longrightarrow V$ is called *normal* if $TT^* = T^*T$, where T^* is the adjoint operator of T. Show that T is normal if and only if $< Tv, Tw >=< T^*v, T^*w >$ for all $v,\ w \in V$.
(c) Assume T is normal. Prove that if v is an eigenvector of T with eigenvalue λ, then v is also an eigenvector of T^* with eigenvalue $\overline{\lambda}$. (3+6+6)